For lines with very low characteristic impedance ($Z_0 \to 0$), the center plate occupies an increasingly larger proportion of the rectangular width, a of the structure. For this case, the center strip coupling becomes large so that $b' \gg b$ and $a/b' \to 0$. Hence, from (4), it is evident that, for this case, $2a/\lambda_{c(01)} \to 0$ for all a/b. Similarly, for the cases of the TE_{11} and TE_{21} modes, it can be seen that $2a/\lambda_{c(11)} \to 1$ and $2a/\lambda_{c(21)} \to 2$ for all a/b. Both Gruner's and Baier's results for the rectangular coaxial line [5], [6] confirm the above.

Referring back to Fig. 3 again, it is apparent that, for the only altered TM-mode shown (TM_{11}), the cutoff frequency is increased. Whereas when no center conductor is present (waveguide case), the TM_{11} mode will always propagate before the TE_{21} mode, this situation generally becomes reversed when the center strip is present. For the case of a 50-Ω line, it is apparent that the TE_{21} cutoff is below that of the TM_{11} mode for all $a/b \geqslant 0.9$. Note that the presence of a relatively narrow center strip ($w/a < 0.2$) causes a marked increase in the TM_{11} cutoff, but that this increase does not exceed that corresponding to the TM_{12} cutoff. In fact, for lines with $Z_0 < \sim 70$ Ω, the TM_{11} cutoff is essentially the same as that for the TM_{12} mode. In this case, when the center conductor occupies an appreciable fraction of the width ($0.6a$ or more), it apparently acts as an electrical wall, causing the TM_{11} mode field structure to break up into a TM_{12} structure that contains an H-field null along the x-axis. These results are confirmed by Gruner's data [5] which show the curves for the TM_{11} and TM_{12} cutoff, as well as those for the TM_{21} and TM_{22} modes merging for values of $w/a > 0.6$.

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Field Patterns and Resonant Frequencies of High-Order Modes in an Open Resonator

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***Abstract*—Using the electromagnetic perturbation theory, it is shown that the linearly polarized TEM_{pl} modes ($l > 0$) predicted by conventional methods are not the resonant modes in an open resonator. Instead, two other series of high-order modes are proposed with improved accuracy in resonant frequencies.**

I. Nomenclature

c	velocity of light,
D	distance of separation between reflectors,
$\bar{E}$	electric field strength,
E_x	$\psi(\rho, \theta, z) \cdot \exp(-jkz)$,
f	resonant frequency,
j	$\sqrt{-1}$,
k	propagation constant in free space,
l	azimuthal mode number,
$L_p^l(x)$	generalized Laguerre polynomial,
$L_p^{l\prime}(x)$	$(d/dx) L_p^l(x)$,
p	radial mode number,
q	axial mode number,
R	radius of curvature of the phase front,
R_1	radius of curvature of the reflector,
w	radius of the beam wave,
w_0	radius of the beam waist,
w_1	radius of the beam wave at $z = D/2$,
W	energy stored,
$\hat{z}$	unit vectors along the z direction,
Δ	small increment,
ρ, θ, z	cylindrical coordinates,
$\Phi = \arctan(z/z_0)$	additional phase shift.

II. Introduction

From the approximate beam-wave theory [1], there exists a complete set of linearly polarized Gaussian beam modes, which are conventionally designated as TEM_{pl}. These modes can be separated into two series, and can be represented by

$$E_x = \left(\sqrt{2}\,\frac{\rho}{w}\right)^l L_p^l\left(\frac{2\rho^2}{w^2}\right)\frac{w_0}{w}\exp\left(\frac{-\rho^2}{w^2}\right) \cdot \exp\left[-jkz + j(2p+l+1)\Phi - j\frac{k\rho^2}{2R}\right]\cos l\theta \quad (1)$$

and

$$E_x = \left(\sqrt{2}\,\frac{\rho}{w}\right)^l L_p^l\left(\frac{2\rho^2}{w^2}\right)\frac{w_0}{w}\exp\left(\frac{-\rho^2}{w^2}\right) \cdot \exp\left[-jkz + j(2p+l+1)\Phi - j\frac{k\rho^2}{2R}\right]\sin l\theta \quad (2)$$

where p and l are the radial and azimuthal mode numbers, respectively. By combining two linearly polarized modes of the same order, it is possible to synthesize other polarized modes in

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the form of

$$\left.\begin{aligned} E_x &= \left(\sqrt{2}\,\frac{\rho}{w}\right)^l L_p^l\left(\frac{2\rho^2}{w^2}\right)\frac{w_0}{w}\exp\left(\frac{-\rho^2}{w^2}\right) \\ &\quad\cdot\exp\left[-jkz + j(2p+l+1)\Phi - j\frac{k\rho^2}{2R}\right]\cos l\theta \\ E_y &= \pm\left(\sqrt{2}\,\frac{\rho}{w}\right)^l L_p^l\left(\frac{2\rho^2}{w^2}\right)\frac{w_0}{w}\exp\left(\frac{-\rho^2}{w^2}\right) \\ &\quad\cdot\exp\left[-jkz + j(2p+l+1)\Phi - j\frac{k\rho^2}{2R}\right]\sin l\theta \end{aligned}\right\} \tag{3}$$

and

$$\left.\begin{aligned} E_x &= \left(\sqrt{2}\,\frac{\rho}{w}\right) L_p^1\left(\frac{2\rho^2}{w^2}\right)\frac{w_0}{w}\exp\left(\frac{-\rho^2}{w^2}\right) \\ &\quad\cdot\exp\left[-jkz + j(2p+2)\Phi - j\frac{k\rho^2}{2R}\right]\sin\theta \\ E_y &= -\left(\sqrt{2}\,\frac{\rho}{w}\right) L_p^1\left(\frac{2\rho^2}{w^2}\right)\frac{w_0}{w}\exp\left(\frac{-\rho^2}{w^2}\right) \\ &\quad\cdot\exp\left[-jkz + j(2p+2)\Phi - j\frac{k\rho^2}{2R}\right]\cos\theta. \end{aligned}\right\} \tag{4}$$

It is shown in the Appendix that the linearly polarized modes represented by (1) and (2) can be obtained from appropriate combinations of the modes represented by (3) and (4). Thus, the solutions (3) together with (4) are also complete. The positive and negative signs can be arbitrarily designated for "series A" and "series B" TEM_{pl} modes, respectively, while the circular electric TEM_{p1} modes (4) may be considered as belonging to "series A". Moreover, the resonant frequency for any one of the above resonant modes can be determined from the following unique formula:

$$f = \frac{c}{2D}\left[q + 1 + \frac{2p+l+1}{\pi}\arccos\left(1 - \frac{D}{R_1}\right)\right]. \tag{5}$$

Refinements for this approximate beam-wave theory have been attempted in the literature by several methods. In the paper by Lax, Louisell, and McKnight [2], it is shown that the transverse field components for a beam wave derived from the approximate beam-wave theory is only the zeroth-order terms of the exact field solutions. Particularly, the expression for the first-order field of the generalized Laguerre-Gaussian beam mode of the spherical open resonator is shown explicitly. This beam mode is assumed to have a zeroth-order field component which is transverse and linearly polarized as described in (1) and (2). This first-order field component is found to be longitudinal. We have, however, recently demonstrated [3] both theoretically and experimentally that, in the specific case $p = 0$, only the high-order modes having the zeroth-order transverse field components as described in (3) and (4) can be found in the open resonator, while the modes having a linearly polarized zeroth-order transverse field component, as described in (1) and (2), are in fact not the resonant modes. In this paper, a different approach, based on electromagnetic perturbation theory, will be carried out to show that the linearly polarized TEM_{plq} modes (with $p \geqslant 0$ and $l > 0$) do not exist in the open resonator. (Here, we still adopt the conventional but somewhat inappropriate notation—linearly polarized TEM_{plq} mode. In this paper, this notation is understood to represent the generalized Laguerre–Gaussian beam mode which has a zeroth-order transverse field component.) This finding is more general than our previous result, which is obtained by using the complex-source-point theory and the result also serves as a check on our previous claim [3].

In the two related papers [4], [5], the first-order perturbation calculations have been examined by Erickson in an attempt to improve the accuracy of the resonant formula for the general linearly polarized TEM_{plq} mode. Specifically, the first perturbation is concerned with the neglected term $\partial^2\psi/\partial z^2$ in the scalar wave equation, and the second perturbation is due with a change of the constant phase surface of the approximate solution to that of the spherical reflector. However, Cullen [6] has demonstrated for the fundamental beam mode that, in order to obtain the desired accuracy, a third perturbation result should be added. This perturbation arises from a change of the boundary condition $E_x = 0$ to $E_{\tan} = 0$ over the reflecting surface, $E_{\tan}$ being the tangential component of the electric field.

The objective of this paper is to examine the three perturbation calculations for the general "series A" and "series B" modes described by (3). The result so obtained improves the resonant formula (5) explicitly. And, most importantly, a difference found between the "series A" and "series B" modes of the same order ($l > 0$) provides an argument for the nonexistence of linearly polarized TEM_{plq} modes ($l > 0$) in the open resonator.

II. Perturbation Analysis

A. Perturbation of the Differential Operator

The conventional beam wave formulas are the result of an approximation that the term $\partial^2\psi/\partial z^2$ in the scalar wave equation

$$\nabla^2 E_x + k^2 E_x = 0 \tag{6}$$

is small and to be neglected. A perturbation calculation for this neglected term has been evaluated by Erickson [5] for the linearly polarized modes (1) and (2). It is straight forward to see that his result is applicable to the other polarized modes (3) and (4). The resulting frequency shift is given by

$$\Delta f = \frac{c}{\pi D}\arctan\left[\frac{D}{4k^3 w_0^4}\left(6p^2 + 6pl + l^2 + 6p + 3l + 2\right)\right] \tag{7}$$

which may be further approximated by

$$\Delta f = \frac{c}{2D}\,\frac{1}{4\pi k R_1}\,\frac{w_1^2}{w_0^2}\left(6p^2 + 6pl + l^2 + 6p + 3l + 2\right) \tag{8}$$

with

$$k^2 w_0^2 w_1^2 = 2R_1 D. \tag{9}$$

From (8), we have

$$\frac{\Delta f}{f} = \frac{\left(6p^2 + 6pl + l^2 + 6p + 3l + 2\right)}{2k^4 w_0^4} \tag{10}$$

which gives the order of magnitude of the error inherent in the approximate resonant formula (5) due to the neglecting of the term $\partial^2\psi/\partial z^2$ in the wave equation.

B. Perturbation of Boundary Surface

To calculate the frequency shift due to the deforming of the constant phase surface $\bar{E} = 0$ (both E_x and E_y in (3)) into a spherical shape, the functions Φ and $1/R$ can be expanded in terms of $z - z_1 (z_1 = D/2)$, and then $z - z_1$ can be replaced by the value $-\rho^2/2R(z_1)$. This yields

$$\left.\begin{aligned} \Phi(z) &= \Phi(z_1) - \rho^2/kw_1^2 R(z_1) \\ \frac{k\rho^2}{2R(z)} &= \frac{k\rho^2}{2R(z_1)} + \frac{k\rho^4}{2R^3(z_1)}\left(1 - \frac{R(z_1)}{2z_1}\right) \end{aligned}\right\} \tag{11}$$

both of which are then substituted into (3) to give an equation for the constant phase surface $\bar{E}=0$ as

$$z_N = z_1 - \frac{\rho^2}{2R_1} - \frac{(2p+l+1)\rho^2}{k^2 w_1^2 R_1} - \frac{\rho^4}{2R_1^3}\left(1-\frac{R_1}{2z_1}\right). \quad (12)$$

Here, $R_1 = R(z_1)$, $w_1 = w(z_1)$ and z_N is the z-coordinate on the constant phase surface. Note that (12) is accurate for the range of ρ on which the field strength is of significance. To the same degree of accuracy, the surface of a spherical reflector of radius of curvature R_1 can be expressed as

$$z_S = z_1 - \frac{\rho^2}{2R_1} - \frac{\rho^4}{8R_1^3}. \quad (13)$$

Using the action theorem $\Delta f/f = \Delta W/W$, the required frequency shift can be determined. To do so, we can write

$$\Delta W = 2\int_0^{2\pi}\int_0^{\infty}\frac{1}{4}\mu_0\left[|H_x|^2+|H_y|^2\right]\Delta z \rho\, d\rho\, d\phi \quad (14)$$

with

$$\Delta z = z_N - z_S = -\frac{(2p+l+1)\rho^2}{k^2 w_1^2 R_1} + \left(3-\frac{w_1^2}{w_0^2}\right)\frac{\rho^4}{2k^2 w_1^4 R_1} \quad (15)$$

and

$$|H_y|^2 + |H_x|^2 = |H_0|^2\left(\frac{2\rho^2}{w_1^2}\right) L_p^{l2}\left(\frac{2\rho^2}{w_1^2}\right)\exp\left(\frac{-2\rho^2}{w_1^2}\right). \quad (16)$$

Using the integrals for the Laguerre polynomials [7], we arrive at

$$\Delta W = \frac{\pi\mu_0|H_0|^2 w_1^2 (l+p)!}{32k^2 R_1 p!} \cdot \left[2p^2+2pl-l^2+2p+l+2 - \frac{w_1^2}{w_0^2}(6p^2+6pl+l^2+6p+3l+2)\right]. \quad (17)$$

To the same degree of accuracy, the average energy stored is found to be

$$W = \frac{1}{8}\pi\mu_0|H_0|^2 w_1^2 D\cdot\frac{(l+p)!}{p!}. \quad (18)$$

Thus

$$\Delta f = \frac{c}{2D}\cdot\frac{1}{4\pi k R_1}\left[2p^2+2pl-l^2+2p+l+2 - \frac{w_1^2}{w_0^2}(6p^2+6pl+l^2+6p+3l+2)\right]. \quad (19)$$

C. Perturbation of the Nonvanishing Components of E_x and E_y on the Reflecting Surface

The use of the boundary condition $E_x = E_y = 0$ on the mirror surface is not correct. The reason is that, on the perfectly conducting mirror, the electric field (vector) is only required to be normal to the mirror surface, thus, there may be finite values of E_x and E_y on the surface. Following a similar procedure adopted by Cullen [6], let us suppose u and v are two different representations of E_x and both are solutions of the scalar wave equation, with $u=0$ and $v=v_s$ on the mirror surface S. Then a first-order perturbation formula concerning with the nonvanishing of E_x on the surface can be deduced from the Green's second identity as

$$\frac{\Delta f}{f} = \frac{\int_s (v_s \nabla u)\cdot d\bar{S}}{2k_u^2\int_v u^2\, dV} \quad (20)$$

where the surface integral is taken over both of the two mirrors.

To evaluate (19), v_s must be determined first. Initially, the divergence equation will be used to estimate E_z. In our first-order approximation, the variation of the functions R, Φ, and w with z can be neglected when deriving the expression for E_z. Thus, (3) can be substituted into

$$E_z = \frac{1}{jk}\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}\right) \quad (21)$$

to give

$$\begin{aligned} E_z = {} & j\frac{2\rho}{kw^2}\left(\frac{\sqrt{2}\rho}{w}\right)^l \exp\left(\frac{-\rho^2}{w^2}\right)\cdot\cos(l-1)\theta \\ & \cdot\exp\left\{-j\left[kz-(2p+l+1)\Phi+\frac{k\rho^2}{2R}\right]\right\} \\ & \cdot\left[L_p^l\left(\frac{2\rho^2}{w^2}\right)\exp j\Phi - \frac{2w_0}{w}L_p^{l\prime}\left(\frac{2\rho^2}{w^2}\right)\right] \\ & - j\frac{2\sqrt{2}l}{kw}\left(\frac{\sqrt{2}\rho}{w}\right)^{l-1}\frac{w_0}{w} \\ & \cdot\exp\left(\frac{-\rho^2}{w^2}\right)L_p^l\left(\frac{2\rho^2}{w^2}\right)\cos(l-1)\theta \\ & \cdot\exp\left\{-j\left[kz-(2p+l+1)\Phi+\frac{k\rho^2}{2R}\right]\right\} \end{aligned} \quad (22)$$

for "series A" TEM_{pl} modes, and

$$\begin{aligned} E_z = {} & j\frac{2\rho}{kw^2}\left(\frac{\sqrt{2}\rho}{w}\right)^l \exp\left(\frac{-\rho^2}{w^2}\right)\cdot\cos(l+1)\theta \\ & \cdot\exp\left\{-j\left[kz-(2p+l+1)\Phi+\frac{k\rho^2}{2R}\right]\right\} \\ & \cdot\left[L_p^l\left(\frac{2\rho^2}{w^2}\right)\exp j\Phi - \frac{2w_0}{w}L_p^{l\prime}\left(\frac{2\rho^2}{w^2}\right)\right] \end{aligned} \quad (23)$$

for "series B" TEM_{pl} modes.

By taking the imaginary part of (3), and (22)–(23), the standing wave solutions corresponding to the odd axial modes in an open resonator can be derived as

$$E_x = \left(\sqrt{2}\frac{\rho}{w}\right)^l L_p^l\left(\frac{2\rho^2}{w^2}\right)\frac{w_0}{w}\exp\left(\frac{-\rho^2}{w^2}\right) \cdot\sin\left[kz-(2p+l+1)\Phi+\frac{k\rho^2}{2R}\right]\cos l\theta \quad (24)$$

$$E_y = \pm\left(\sqrt{2}\frac{\rho}{w}\right)^l L_p^l\left(\frac{2\rho^2}{w^2}\right)\frac{w_0}{w}\exp\left(\frac{-\rho^2}{w^2}\right) \cdot\sin\left[kz-(2p+l+1)\Phi+\frac{k\rho^2}{2R}\right]\sin l\theta \quad (25)$$

$$
\begin{aligned}
E_z = & \frac{2\rho}{kw^2}\left(\sqrt{2}\,\frac{\rho}{w}\right)^l L_p^l\left(\frac{2\rho^2}{w^2}\right)\exp\left(\frac{-\rho^2}{w^2}\right) \\
& \cdot\cos\left[kz-(2p+l+2)\Phi+\frac{k\rho^2}{2R}\right]\cos(l\mp1)\theta \\
& -\frac{4\rho}{kw^2}\left(\sqrt{2}\,\frac{\rho}{w}\right)^l L_p^{l'}\left(\frac{2\rho^2}{w^2}\right)\frac{w_0}{w}\exp\left(\frac{-\rho^2}{w^2}\right) \\
& \cdot\cos\left[kz-(2p+l+1)\Phi+\frac{k\rho^2}{2R}\right] \\
& \cdot\cos(l\mp1)\theta-\frac{\sqrt{2}\,l}{kw}\left(\sqrt{2}\,\frac{\rho}{w}\right)^l L_p^l\left(\frac{2\rho^2}{w^2}\right)\frac{w_0}{w} \\
& \cdot\exp\left(\frac{-\rho^2}{w^2}\right)\cos\left[kz-(2p+l+1)\Phi+\frac{k\rho^2}{2R}\right] \\
& \cdot[\cos(l-1)\theta\pm\cos(l-1)\theta]
\end{aligned} \tag{26}
$$

where the upper signs and lower signs are for the "series A" and "series B" TEM_{plq} modes, respectively, (q is the axial mode number).

To determine the value of E_z on the mirror surface at $+z_1$, we can insert, consistent with our first-order approximation

$$
\left.\begin{aligned}
& z = z_1-\rho^2/2R_1 \\
& kz_1-\arctan(z_1/z_0)=(q+1)\pi/2
\end{aligned}\right\} \tag{27}
$$

into (26), and neglect the variation of w, R, and Φ with z to give

$$
\begin{aligned}
E_{z_s} = & \frac{2\rho}{kw_1^2}\left(\frac{\sqrt{2}\,\rho}{w_1}\right)^l(-1)^{(q+1)/2}\frac{w_0}{w_1}\exp\left(\frac{-\rho^2}{w_1^2}\right) \\
& \cdot\left[L_p^l\left(\frac{2\rho^2}{w_1^2}\right)-2L_p^{l'}\left(\frac{2\rho^2}{w_1^2}\right)\right]\cos(l\mp1)\theta \\
& -\frac{\sqrt{2}\,l}{kw_1}\left(\frac{\sqrt{2}\,\rho}{w_1}\right)^{l-1}(-1)^{(q+1)/2}\frac{w_0}{w_1}\exp\left(\frac{-\rho^2}{w_1^2}\right)L_p^l\left(\frac{2\rho^2}{w_1^2}\right) \\
& \cdot[\cos(l-1)\theta\pm\cos(l-1)\theta].
\end{aligned} \tag{28}
$$

Thus, using the condition $E_{\tan}=0$, we find

$$
\begin{aligned}
v_s = E_{x_s} = & E_{z_s}\cdot\frac{x}{R_1} \\
= & \frac{2\rho^2\cos\theta\cos(l-1)\theta}{kw_1^2R_1}\left(\frac{\sqrt{2}\,\rho}{w_1}\right)^{l-2} \\
& \cdot(-1)^{(q+1)/2}\frac{w_0}{w_1}\exp\left(\frac{-\rho^2}{w_1^2}\right) \\
& \cdot\left[2\left(\frac{\rho^2}{w_1^2}-l\right)L_p^l\left(\frac{2\rho^2}{w_1^2}\right)-\frac{4\rho^2}{w_1^2}L_p^{l'}\left(\frac{2\rho^2}{w_1^2}\right)\right]
\end{aligned} \tag{29}
$$

for "series A" TEM_{plq} modes, and

$$
\begin{aligned}
v_s = & \frac{2\rho^2\cos\theta\cos(l+1)\theta}{kw_1^2R_1}\left(\frac{\sqrt{2}\,\rho}{w_1}\right)^l \\
& \cdot(-1)^{(q+1)/2}\frac{w_0}{w_1}\exp\left(\frac{-\rho^2}{w_1^2}\right) \\
& \cdot\left[L_p^l\left(\frac{2\rho^2}{w_1^2}\right)-2L_p^{l'}\left(\frac{2\rho^2}{w_1^2}\right)\right]
\end{aligned} \tag{30}
$$

for "series B" TEM_{plq} modes.

Moreover, to the same degree of approximation, we have

$$
\nabla u = -\frac{kw_0}{w_1}\left(\frac{\sqrt{2}\,\rho}{w_1}\right)^l L_p^l\left(\frac{2\rho^2}{w_1^2}\right)\exp\left(\frac{-\rho^2}{w_1^2}\right)(-1)^{(q+1)/2}\cos l\theta\hat{z} \tag{31}
$$

on the mirror surface.

Now, using (29), (30), and (31), and retaining only first-order terms, the perturbation formula (20) can be evaluated to yield

$$
\frac{\Delta f}{f} = \begin{cases} \dfrac{-(1-l)}{k^2R_1D}, & \text{(for "series } A\text{")} \\[2ex] \dfrac{-(1+l)}{k^2R_1D}, & \text{(for "series } B\text{")} \end{cases} \tag{32}
$$

or

$$
\Delta f = \begin{cases} -\dfrac{c}{2D}\cdot\dfrac{(1-l)}{\pi kR_1}, & \text{(for "series } A\text{")} \\[2ex] -\dfrac{c}{2D}\cdot\dfrac{(1+l)}{\pi kR_1}, & \text{(for "series } B\text{").} \end{cases} \tag{33}
$$

We would like to point out that, when using the divergence equation (21), the azimuthal mode number l is assumed not to be zero ($E_\theta \neq 0$), thus, (33) can only be considered as a good estimate of the frequency shift for the resonant modes that $l\neq0$, but not for the pure radial modes. The argument can be referred to Cullen [6].

IV. Conclusion

It is reasonable to add all the three perturbed frequency shifts (8), (19), and (33) to give

$$
\Delta f = \begin{cases} \dfrac{c}{2D}\cdot\dfrac{1}{4\pi kR_1}\cdot(2p^2+2pl-l^2+2p+5l-2) \\ \qquad\qquad\text{(for "series } A\text{")} \\[2ex] \dfrac{c}{2D}\cdot\dfrac{1}{4\pi kR_1}\cdot(2p^2+2pl-l^2+2p-3l-2) \\ \qquad\qquad\text{(for "series } B\text{").} \end{cases} \tag{34}
$$

It can also be shown that (34) is also true for the even axial modes and the circular electric TEM_{p1} modes (4). As a check, we may put $p=0$ in (34) which will become the final term of (10) in [3].

To conclude, there exists a difference in the resonant frequency between "series A" and "series B" TEM_{plq} modes ($l>0$) of the same order in the "large aperture" spherical open resonator. Therefore, they cannot be superimposed to produce the linearly polarized TEM_{plq} modes. Also, we have obtained an improvement in the accuracy of the resonant formula for both series of the high-order modes, at least for $l\neq0$.

Appendix

Construction of the Linearly Polarized Modes from the "Series A" and the "Series B" Modes

In this appendix, we shall show that the linearly polarized modes can be obtained by linear combinations of the "series A" and the "series B" modes from the point of view of the conventional beam-wave theory.

First, it is obvious that the linearly polarized modes described in (1) can be deduced by adding together the "series A" and the "series B" modes described in (3).

Next, in order to obtain the linearly polarized modes described in (2) from the modes described in (3) and (4), the technique of transformation of coordinate systems and vector components may be adopted.

Let us consider the rotation of the x- and y-axis about the origin to an angle β counterclockwise. Mathematically, we transform the (ρ, θ) coordinates to the (ρ^*, θ^*) coordinates, where

$$\begin{cases} \theta^* = \theta - \beta \\ \rho^* = \rho. \end{cases} \tag{A1}$$

Then the transformation of the field components can be obtained by

$$\begin{bmatrix} E_x^*(\rho^*, \theta^*) \\ E_y^*(\rho^*, \theta^*) \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} E_x(\rho, \theta) \\ E_y(\rho, \theta) \end{bmatrix}. \tag{A2}$$

Substituting (3) into (A2), we have

$$\begin{cases} E_x^* = A^* \cos[l\theta^* + (l \mp 1)\beta] \\ E_y^* = \pm A^* \sin[l\theta^* + (l \mp 1)\beta] \end{cases} \tag{A3}$$

with

$$A^* = \left(\sqrt{2}\,\frac{\rho^*}{w}\right)^l L_p^l\left(\frac{2\rho^{*2}}{w^2}\right)\frac{w_0}{w}\exp\left(-\frac{\rho^{*2}}{w^2}\right) \cdot \exp\left[-jkz + j(2p+l+1)\Phi - j\frac{k\rho^{*2}}{2R}\right]. \tag{A4}$$

Now, consider that the "series A" mode is rotated to $\beta = \pi/2(l-1)$, (A3) is reduced to

$$\begin{cases} E_x = -A\sin l\theta \\ E_y = A\cos l\theta. \end{cases} \tag{A5}$$

These expressions are true for $l > 1$. The superscript * is dropped from now on. On the other hand, consider that the "series B" mode is rotated to $\beta = \pi/2(l+1)$, (A3) is reduced to

$$\begin{cases} E_x = -A\sin l\theta \\ E_y = -A\cos l\theta. \end{cases} \tag{A6}$$

These expressions are true for $l \geqslant 1$. Combining (A5) and (A6), we find

$$\begin{cases} E_x = -2A\sin l\theta \\ E_y = 0 \end{cases} \tag{A7}$$

which is identical to (2), apart from a constant factor. It appears that these expressions are only true for $l > 1$ but, using (4) and (A6), it is obvious that (A7) is also true for the special case $l = 1$.

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Phased-Dipole Applicators for Torso Heating in Electromagnetic Hyperthermia

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Abstract —**The paper describes a two-dipole applicator that is capable of providing in-depth and relatively uniform rates of heating (SAR's) over the volume of the torso and greatly reduced SAR's for the rest of the body. Power coupling efficiencies in excess of 60 percent and fairly low leakage power densities have been measured for the applicator.**

I. Introduction

Hyperthermia is considered to be a potentiator of radiation therapy or chemotherapy for many forms of cancer [1],[2]. Among the various techniques, such as conventional heating, or heating by ultrasonic or electromagnetic energy, the latter offers the advantage of minimal reflections at interfaces with bones or with air cavities. Because of the somewhat shallower depth of penetration of electromagnetic energy (on the order of 5–10 cms), phased-array applicators have, however, had to be used to obtain in-depth heating at the tumor sites. In our previous work [3], we have proposed and provided theoretical designs of applicators consisting of short dipoles which may be altered in position and magnitude and phase of excitation for each of its elements so as to obtain minimum deviation from prescribed inhomogeneous rates of heating (SAR's) for the various parts of the body. Recognizing that designs for multidipole applicators for prescribed temperature distributions would be of greater interest, we have also recently started to develop an inhomogeneous thermal model of man to allow for inhomogeneities of tissue electrical and thermal properties and for increase in blood flow rates due to vasodilation at elevated temperatures [4]. This paper gives the experimental results obtained with scale models on phased-dipole applicators for torso heating. Given here are the efficiencies for whole-body coupling, the SAR distributions over the volume of the torso and elsewhere within the body, and the strength of the leakage fields from the cylindrical metal casing.

II. Phased-Dipole Applicators

A conceptual illustration of the multidipole applicator is shown in Fig. 1. The applicator uses short dipoles (of lengths less than or equal to $0.1 \times$ wavelength) whose respective positions and excitations (magnitude and phase) are obtained on the basis of numerical calculations with a block model of man [5] for minimum deviation from prescribed inhomogeneous SAR's for the various parts of the body. A metal cylinder, which may, of course, be constructed of metal screening, helps to contain the fields to the absorber that is the human body. The radius of the cylinder is not critical but image theory must be used to correct for the cylinder in numerical calculations. Of the various designs presented in [3], the one used for the present experiments is the applicator design for abdominal heating. For this application, one dipole placed ventrally in the symmetrical plane at a radial distance of 0.35 m and at a location of 1.0 m above the base of the feet was found to be adequate to give SAR's in the abdominal volume that were three times or more than those for the rest of

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